DHANAMANJURI UNIVERSITY

Examination- 2025 (June)

M.A/M.Sc. 2nd Semester

Name of Programme : M.A/M. Sc. Mathematics
Paper Type : Theory
Paper Code : MAT-506
Paper Title : Advanced Abstract Algebra-II
Full Marks : 80
Pass Marks : 32

Duration: 3 Hours

The figures in the margin indicate full marks for the questions

Answer all the questions:

Answer any three from the following questions: $10 \times 3 = 30$

1. Let R be a ring and M be an $R-module$. Then for any $x \in M$, show that the set $K = [rx + nx | r \in R, n \in \mathbb{Z}]$ is the smallest $R - submodule$ of M containg x. Further, show that $K = Rx$, if R has unity.

2. Let R be a ring with unity, and let M be an $R - module$. Then prove that the following are equivalent

   a) M is simple

   b) $M \neq (0)$ and M is generated by and $0 \neq x \in M$.

   c) $M \cong \frac{R}{I}$, where I is a maximal left ideal of R.

3. a) Is it true that every module has a basis? Justify it. 5

   b) Let M be a free $R - module$ with a basis $\{e_1, e_2, \dots e_3\}$. Show that $M \cong R^n$. 5

4. For any $R - module$ M and an exact sequence

   $0 \to N' \xrightarrow{f} N \xrightarrow{g} N^{"} \to 0$, prove that the induced sequence

   $0 \to Hom_R(M, N') \xrightarrow{f_*} Hom_R(M, N) \xrightarrow{g_*} Hom_R(M, N^{"})$ is exact.

5. Let $M = \sum_{i=1}^{k} M_i$ be a direct sum of modules $M_i, 1 \leq i \leq k$. Then prove that the ring $Hom_R(M, M)$ is isomorphic to

$$\begin{bmatrix} Hom_R(M_1, M_1) & Hom_R(M_2, M_1) & Hom_R(M_k, M_1) \\ Hom_R(M_1, M_k) & Hom_R(M_2, M_k) & Hom_R(M_k, M_k) \end{bmatrix}$$

Answer any three from the following questions: $10 \times 3 = 30$

6. Let M be an $R-module$ and let N be an $R-submodule$ of M. Then, show that M is Artinian iff both N and $\frac{M}{N}$ are Artinian.

7. a) Let A be a minimal left ideal in a ring R. Then show that either $A^2 = (0)$ or $A = Re$, where e is an idempotent in R.

   b) Let R be Noetherian, the show that each ideal contains a finite product of prime ideals.

8. Let R be a principal ideal domain with unity, and F be a free $R-module$ with a basis consisting of n elements. Then show that any submodule K of F is also free with a basis consisiting of m elements such tha $m \leq n$.

9. For an $R-module$ M, prove that the following are equivalent.

   a) M is Noetherain.

   b) Every submodule of M is finitely generated.

   c) Every non empty set S of submodules of M has a maximal element.

10. Obtain the Smith Normal form and invariant factors of the matrix A over $\mathbb{Q}[x]$, where the matrix A is given by

$$\begin{bmatrix} 5-x & 1 & -2 & 4 \\ 0 & 5-x & 2 & 2 \\ 0 & 0 & 5-x & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Answer any two from the following: $10 \times 2 = 20$

11. Let R be a PID, and M be any finity generated $R-module$. Then prove that $M \cong R^s \oplus \frac{R}{Ra_1} \dots \oplus \frac{R}{Ra_n}$, a direct sum of cyclic module where a_i are non-zero, non-units and $a_i | a_i + 1, i = 1, \dots, r-1$.
12. Determine all the possible Jordan canonical forms for the operator $T: V \to V$ whose characteristics polynomial is $(x-2)^3(x-5)^2$.
13. Define rational canonical form of a linear transformation. Also find the rational canonical form on a space of dimension 6 over $\mathbb{R}$ with minimal polynomial $(x^2+1)(x-5)^2$.